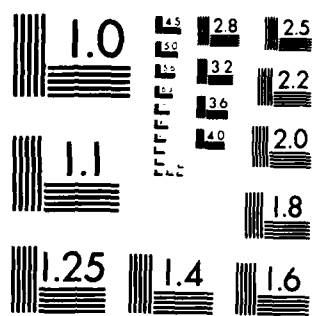MICROCOPY RESOLUTION TEST CHART
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DYNAMIC CHANGES OF PHASE IN A
VAN DER WAALS FLUID

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ABSTRACT

This paper gives sufficient conditions to guarantee the existence of a

shock layer solution connecting two different eouilibrium states in a van der

Waals fluid. In particular, the equilibrium states can belong to two

different phases of the fluid. The constitutive laws come from a modified

Korteweg theory which is compatible with the Clausius Duhem inequality. The

Clausius Duhem inequality in turn gives rise to a Liapunov function. The main

mathematical tool is the LaSalle invariance principle.

SIGNIFICANCE AND EXPLANATION

In the study of both static and dynamic phase transitions a typical model
has the constitutive law for stress given by Korteweg's theory. In this
theory, the classical form of Navier and Stokes has additional terms which
model the effects of interfacial capillarity and depend on the spatial
gradients of the density up to second order. It is these capillarity terms
which allow for smooth static phase transitions. Unfortunately, Korteweg's
theory is not compatible with the usual continuum theory of thermodynamics
since the Clausius-Duhem inequality is, in general, not satisfied. Recently,
a modified Korteweg theory has been developed by E. Dunn and J. Serrin, that
does not suffer from this defect. We use this theory to generate a set of
shock layer equations which are appropriate for shocks, including dynamic
phase transitions. One of the immediate benefits obtained from this theory is
that the Clausius-Duhem inequality gives rise to a Liapunov function for the
shock layer equations. We then give sufficient conditions for the existence
of a shock layer connecting two equilibrium points.

DYNAMIC CHANGES OF PHASE IN A VAN DER WAALS FLUID

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INTRODUCTION

In a paper by Serrin [1], Korteweg's theory of capillarity [2,3] was used to find
conditions for equilibrium between liquid and vapor phases of a van der Waals fluid. In a
subsequent paper, Slemrod [4] extended Serrin's approach to study dynamic changes of phase
in a van der Waals fluid, under the assumption of isothermal motion. This study was
further extended by Hagan and Slemrod in [5]. The next logical step was to drop the
assumption of isothermal motion. This was done for a van der Waals fluid in a paper [6] by
Slemrod. In [6] he showed the existence of a shock layer that converts vapor to liquid and
the existence of a shock layer that converts liquid to vapor, under assumptions that render
the motion nearly isothermal. These assumptions are that the specific heat capacity at
constant volume is large, the coefficients of heat conduction and viscosity are of the same
small order μ, and the coefficients in the capillarity terms of the stress are of order
μ^2.

One of the problems that complicates the study of dynamic changes of phase is the
incompatibility of the classical Korteweg stress with the Clausius Duhem inequality [7].
Recently, however, a modified Korteweg theory has been developed by Dunn and Serrin [8]
that is compatible with the Clausius Duhem inequality. In this theory they posit the
existence of a rate of supply of mechanical energy, the interstitial working, which takes
into account the working of longer range interactions. With this additional term in the
energy balance it is possible to derive a constitutive relation for stress that depends on
spatial gradients of the density and still satisfies the Clausius Duhem inequality. These
spatial gradient terms are used to model the effects of interfacial capillarity and at the
same time allow the existence of static phase transitions [1], [9]. That is, if we were to
use the classical form of Navier and Stokes for the stress then we would find that some
dynamic phase transitions exist, but no static ones [10], [11].

In this paper we use a special form of the modified Korteweg theory contained in [8]. The Clausius Duhem inequality then gives a direct proof of the increase of entropy across a shock layer and it also provides a Liapunov function, in the sense of LaSalle [12], for the shock layer equations.

In section two of this paper we derive the shock layer equations and the increase of entropy theorem. In section three we examine some of the properties of the Hugoniot curves for a van der Waals fluid. In section four we state sufficient conditions to guarantee the existence of a compressive shock layer in a van der Waals fluid.
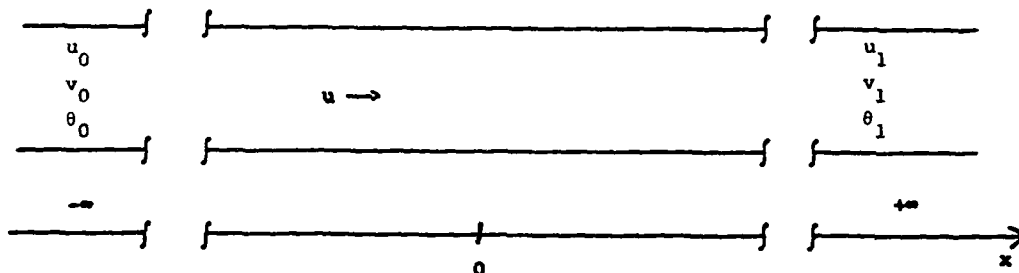
SECTION 2

Figure 1

We consider one-dimensional steady flow of a van der Waals fluid. The flow may be thought of as taking place in a cylinder of uniform cross-section parallel to the x-axis. We will always assume that the fluid velocity u is positive so that the fluid flows from left to right. The <u>absolute temperature</u> will be denoted by θ, and the <u>specific volume</u> by v.

We will seek a smooth solution $(u(x),v(x),\theta(x))$, $x \in R$ of the equations of motion satisfying the following boundary conditions,

(2.1)
$$(u(x),v(x),\theta(x)) \to \begin{cases} (u_0,v_0,\theta_0) & \text{as } x \to -\infty \\ (u_1,v_1,\theta_1) & \text{as } x \to +\infty \end{cases}$$

$(u(x)',v(x)',\theta(x)') \to (0,0,0)$ as $x \to \pm\infty$, and $v''(x) \to 0$ as $x \to \pm\infty$. If such a smooth solution exists, it is called a <u>shock layer</u>. Of particular interest to us in this paper, is the question of the existence of a shock layer, when (v_0,θ_0) and (v_1,θ_1) belong to different phases of a fluid. If such a shock layer exists it will be called a <u>dynamic phase transition</u>.

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The balance laws for mass, momentum, and energy for a one-dimensional steady flow, the absence of external body forces and radiant heating are (see [8])

(2.2)
$$(v^{-1}u)' = 0 ,$$

(2.3)
$$v^{-1}uu' = T'_{xx} ,$$

(2.4)
$$v^{-1}ue' = T_{xx}u' + k' + q',$$

where $( )' = d( )/dx$. In addition to the balance laws we also have the Clausius Duhem inequality,

(2.5)
$$v^{-1}u\eta' > \left(\frac{q}{\theta}\right)' .$$

Here, T_{xx} is the x component of stress in the x direction, e is the specific internal energy, η is the specific entropy, k is the interstitial working, and q is the heat flux.

We will have need of the following notation, if f is a function of v and θ, then $f_1 = f(v_1,\theta_1)$, $f_0 = f(v_0,\theta_0)$, and $[f] = f_1 - f_0$.

Theorem 1.1. (The increase of entropy across a shock layer.)

Suppose $q = 0$ whenever $\theta' = 0$. Then $\eta_1 > \eta_0$ for any shock layer.

Proof.

Integration of (2.2) gives

(2.7)
$$u(x) = mv(x),$$

for some constant m. Furthermore $m > 0$ since $v > 0$, and $u > 0$ by assumption. We may now integrate (2.5) from $x = -\infty$ to $x = +\infty$, to obtain,

$$m(\eta_1 - \eta_0) > \frac{q}{\theta}\Big|_{-\infty}^{\infty} .$$

But $\theta' = 0$ at $x = \pm\infty$, hence

(2.8)
$$\eta_1 > \eta_0 .$$

In addition to the balance laws and the Clausius Duhem inequality we need to specify the constitutive structure of the fluid. We shall use the following special form of a modified Korteweg theory developed by J. E. Dunn and J. Serrin [8]. The two main features of this theory that makes it useful for work in phase transions are the preservation of the Clausius Duhem inequality and the occurrence of higher spatial derivatives in the constitutive relation for stress.

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The constitutive relations are (specialized to one-dimensional flow from equations (1.26) and (1.27) of [8])

$$(2.9) \qquad k = \frac{\sigma u (v')^2}{v} ,$$

$$(2.10) \qquad T_{xx} = -p + (\lambda + 2\mu)u' - \frac{\partial \sigma}{\partial v} \frac{(v')^2}{2} - \frac{\partial \sigma}{\partial \theta} v'\theta' - \sigma v'' ,$$

$$(2.11) \qquad q = \kappa \theta' ,$$

$$(2.12) \qquad \psi = \tilde{\psi} + \frac{\sigma}{2} (v')^2 , \quad \text{the specific Helmholtz free energy,}$$

$$(2.13) \qquad \eta = \tilde{\eta} - \frac{1}{2} \frac{\partial \sigma}{\partial \theta} (v')^2 ,$$

$$(2.14) \qquad e = \tilde{e} + \frac{1}{2} \left(\sigma - \theta \frac{\partial \sigma}{\partial \theta} \right)(v')^2 ,$$

where $\sigma = cv^{-3}$ and c is the surface tension coefficient. Here $\tilde{\eta}$ and $\tilde{e}$ are the equilibrium entropy and internal energy. They are functions of v and θ only, as is the pressure p. Furthermore, these quantities satisfy the classical Maxwell relationships of thermodynamics. In general $\sigma, \lambda, \mu, \kappa$ are positive continuous functions of v and θ, while $\sigma, p, \tilde{e},$ and $\tilde{\eta}$ are C^1.

In this paper we shall restrict ourselves to the case where p and $\tilde{e}$ satisfy the following hypotheses,

$$(H1) \qquad p(v,\theta) = \frac{R\theta}{v - b} - \frac{a}{v^2} , \quad \text{for all } (v,\theta) \in \Omega ,$$

$$(H2) \qquad c_v = \frac{\partial \tilde{e}}{\partial \theta} (v,\theta) > 0, \quad \text{for all } (v,\theta) \in \Omega ,$$

where Ω is defined as $\{(v,\theta) : b < v < \infty, \ 0 < \theta < \infty\}$. Here $R, b,$ and a are positive constants and c_v is the specific heat at constant volume. (H1) is of course the van der Waals equation of state.

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Let us substitute (2.7) into (2.3) and (2.4) to obtain,

(2.15)
$$mu' = T'_{xx} ,$$

(2.16)
$$me' = (T_{xx}u)' - T'_{xx}u + k' + q' .$$

We can now put (2.15) in (2.16) to obtain,

(2.17)
$$m(e + \frac{1}{2} u^2)' = (T_{xx}u)' + k' + q' .$$

Now we may integrate (2.15) and (2.17) from $-\infty$ to x and apply the boundary condition

(2.1). This gives

(2.18)
$$m(u - u_0) = T_{xx} - T_{xx_0} ,$$

(2.19)
$$m(e - \tilde{e}_0 + \frac{1}{2} (u^2 - u_0^2)) = T_{xx}u - T_{xx_0}u_0 + k + q .$$

We now insert (2.7), (2.9), (2.10), (2.11) and (2.14) into (2.18) and (2.19) to obtain,

(2.20)
$$\sigma v'' + \frac{\partial \sigma}{\partial \theta} v'\theta' + \frac{\partial \sigma}{\partial \nu} \frac{(v')^2}{2} - m(\lambda + 2\mu)v' + p - p_0 + m^2(v - v_0) = 0 ,$$

and

(2.21)
$$\kappa\theta' = m\{-\frac{1}{2} \frac{\partial\theta\sigma}{\partial\theta} (v')^2 + \tilde{e} - \tilde{e}_0 + p_0(v - v_0) - \frac{m^2}{2} (v - v_0)^2\} .$$

It is convenient for us to define the following two functions,

(2.22)
$$L(v,\theta) = p(v,\theta) - p_0 + m^2(v - v_0) ,$$

and

(2.23)
$$M(v,\theta) = \tilde{e}(v,\theta) - \tilde{e}_0 + p_0(v - v_0) - \frac{1}{2} m^2(v - v_0)^2 .$$

We may now write (2.20) and (2.21) as a system of three first order ordinary differential equations, namely

(2.24a)
$$v' = w ,$$

(2.24b)
$$\sigma w' + \frac{\partial \sigma}{\partial \theta} w\theta' + \frac{\partial \sigma}{\partial v} \frac{w^2}{2} - m(\lambda + 2\mu)w + L(v,\theta) = 0 ,$$

(2.24c)
$$\kappa\theta' = -\frac{m}{2} \frac{\partial\theta\sigma}{\partial\theta} w^2 + mM(v,\theta), \quad (v,\theta) \in \Omega .$$

We shall refer to this system as the shock layer equations.

Lemma 1.2.

Given (u_0, v_0, θ_0) and (u_1, v_1, θ_1) with (v_0, θ_0), $(v_1, \theta_1) \in \Omega$, a shock layer exists satisfying (2.1) if and only if there exist a solution to (2.24) satisfying,

(2.25)
$$(v(x), w(x), \theta(x)) \to \begin{cases} (v_0, 0, \theta_0) & \text{as } x \to -\infty \\ (v_1, 0, \theta_1) & \text{as } x \to \infty, \end{cases}$$

and (2.7) is satisfied.

Lemma 1.3. (The Rankine-Hugoniot jump conditions).

A necessary condition for a shock layer to exist satisfying (2.1) is that the Rankine-Hugoniot jump conditions are satisfied:

(2.26a)
$$[u] = m[v],$$

(2.26b)
$$[p] + m^2[v] = 0,$$

(2.26c)
$$\widetilde{[e]} + \frac{p_1 + p_0}{2} [v] = 0.$$

Proof.

If a shock layer exists satisfying (2.1) then there exists a solution of (2.24) satisfying (2.25) and (2.7). Let $x \to \infty$; then we have from (2.24b) and (2.24c)

(2.27)
$$M_1 = L_1 = 0;$$

hence $[p] + m^2[v] = 0$ and $\widetilde{[e]} + p_0[v] - \frac{1}{2} m^2[v]^2 = 0$. Therefore $\widetilde{[e]} + \frac{1}{2} (p_1 + p_0)[v] = 0$. Now let $x \to \pm\infty$ in (2.7); then $u_1 = mv_1$ and $u_0 = mv_0$. Therefore $[u] = m[v]$.

We can solve the van der Waals equation of state for θ in terms of v and p algebraically. Hence we can define $\hat{e}(v, p) = \widetilde{e}(v, \theta)$. We may then define

(2.28)
$$H(v, p; v_0, p_0) = \hat{e}(v, p) - \hat{e}(v_0, p_0) + \frac{p + p_0}{2} (v - v_0).$$

The curve in the $v - p$ plane consisting of all (v, p) satisfying $H(v, p; v_0, p_0) = 0$ is called the Hugoniot curve generated by (v_0, p_0). Let us note that any state (v_1, p_1) lying in the intersection of the Hugoniot curve with the straight line given by $p = p_0 - m^2(v - v_0)$ will satisfy (2.26b,c) or equivalently (2.27). The corresponding jump $[u]$ is then given directly by (2.26a). Thus all conditions of (2.26) are satisfied.

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It is convenient at this point to group together several thermodynamic identities which will be useful in the following sections. First, we have the standard Gibbs identity

$$(2.29) \qquad \theta d\tilde{\eta} = d\tilde{e} + p dv$$

and the Maxwell relations

$$(2.30) \qquad \frac{\partial \tilde{e}}{\partial v} = \theta \frac{\partial p}{\partial \theta} - p$$

$$(2.31) \qquad \frac{\partial \tilde{\eta}}{\partial v} = \frac{\partial p}{\partial \theta} .$$

In addition, we shall need the formula

$$(2.32) \qquad \frac{\partial p}{\partial v}\Big|_{\tilde{\eta}} = \frac{\partial p}{\partial v} - \frac{\theta}{c_v} \left(\frac{\partial p}{\partial \theta}\right)^2 ,$$

which follows from the chain rule

$$\frac{\partial p}{\partial v} = \frac{\partial p}{\partial v}\Big|_{\tilde{\eta}} + \frac{\partial p}{\partial \tilde{\eta}}\Big|_v \frac{\partial \tilde{\eta}}{\partial v}$$

together with (2.31) and the relation

$$\frac{\partial p}{\partial \tilde{\eta}}\Big|_v = \frac{\partial p}{\partial \theta} \Big/ \frac{\partial \tilde{\eta}}{\partial \theta} = \frac{\theta}{c_v} \frac{\partial p}{\partial \theta} .$$

From (2.30) one also gets $\partial \tilde{e}/\partial v = a/v^2$ for the van der Waals equation of state, so that in this case

$$(2.33) \qquad \tilde{e} = -\frac{a}{v} + \int c_v(\theta) d\theta .$$

SECTION 3

The van der Waals equation of state possesses non-monotone isotherms for $\theta < \theta_c$, where θ_c is the critical temperature and monotone decreasing isotherms for $\theta > \theta_c$. (See Figure 2). We will define the unstable region in the $v - \theta$ plane as

(3.1)
$$\Omega_u = \left\{ (v,\theta) \ \epsilon \ \Omega : \frac{\partial p}{\partial v} (v,\theta) > 0 \right\} .$$

Thus Ω_u is bounded by

(3.2)
$$\theta = \theta_u(v) \equiv \frac{2a}{R} \frac{(v - b)^2}{v^3}, \quad b < v < \infty .$$

We have $\dfrac{d\theta_u(v)}{dv} = \dfrac{-2a(v - b)(v - 3b)}{Rv^4}$ and hence $\theta_u(v)$ has only one stationary point $v_c = 3b$ on (b,∞). Clearly, $\theta_u(v)$ takes on its maximum value $\theta_c = 8a/27Rb$, at $v = v_c$.
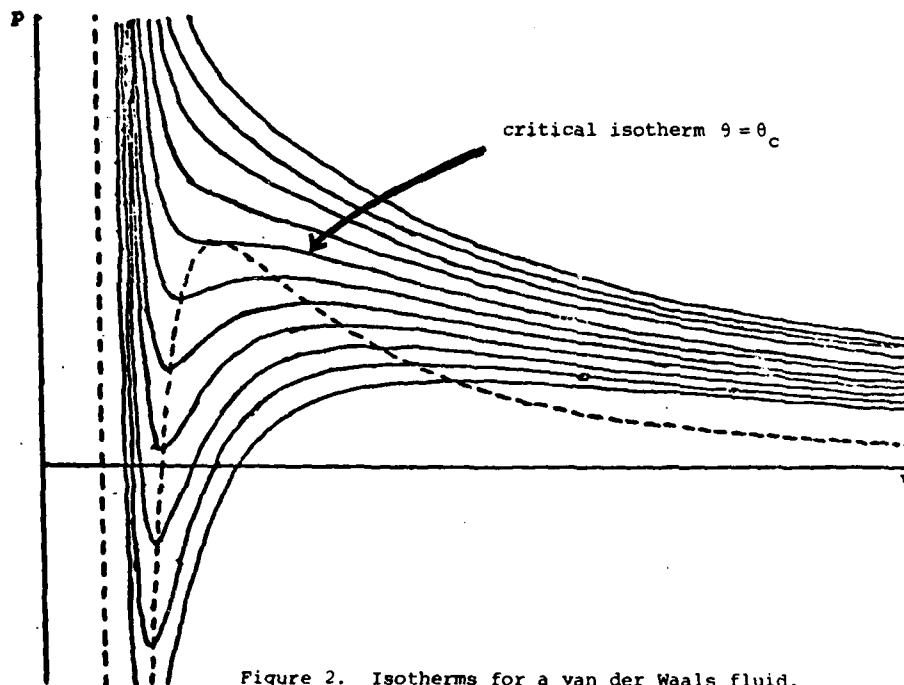

Figure 2. Isotherms for a van der Waals fluid.

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We now define

$$(3.3) \qquad \Omega_s = \{(v,\theta) \in \Omega : \theta_c < \theta < \infty\}$$

$$(3.4) \qquad \Omega_\ell = \{(v,\theta) \in \Omega : \theta_u(v) < \theta < \theta_c, \ b < v < v_c\}$$

$$(3.5) \qquad \Omega_\upsilon = \{(v,\theta) \in \Omega : \theta_u(v) < \theta < \theta_c, \ v_c < v < \infty\}$$

Here Ω_s is the super-heated vapor region, Ω_ℓ is the liquid region, and Ω_υ is the vapor region. (See Figure 3).
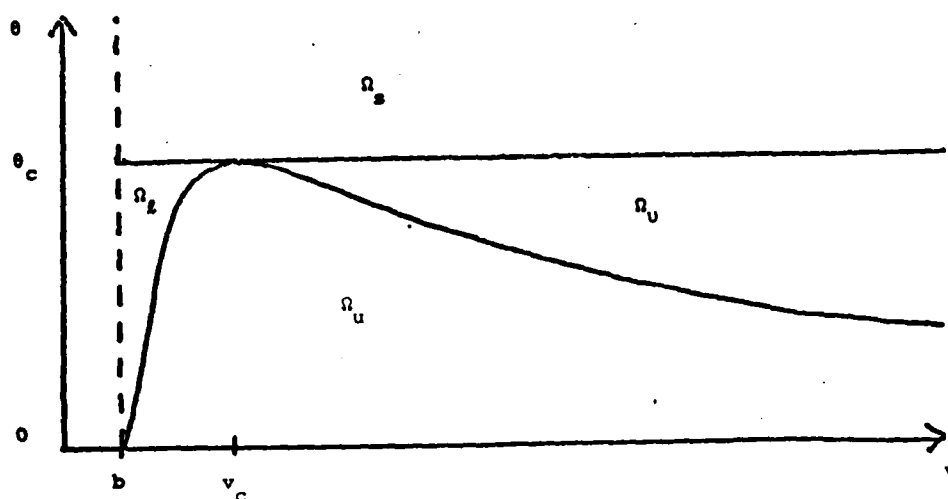
Figure 3

We shall on occasion refer to the fluid state in terms of v and p instead of v and θ. On such occasions it is useful to define the notation

$$\hat{\Omega}_\chi = \{(v,p(v,\theta)) : (v,\theta) \in \Omega_\chi\} \quad \text{where } \chi \text{ is } \ell,s,u,\upsilon, \text{ or empty.}$$

Note that the map $(v,\theta) \to (v,p(v,\theta))$ is a homeomorphism of Ω onto $\hat{\Omega}$.

We now state and prove some useful lemmas concerning the Hugoniot curve.

Lemma 3.1.

Let $(v_0,p_0) \in \hat{\Omega}$ and let I be any compact interval such that $v_0 \in I \subseteq (b,\infty)$. Then the Hugoniot curve generated by (v_0,p_0) approaches the isotherm passing through (v_0,p_0) uniformly on I as $\inf_{0<\theta<\infty} c_v(\theta)$ approaches infinity.

-10-

Proof.

$$H(v,p;v_0,P_0) = \hat{e}(v,p) - \hat{e}_0 + \frac{p + p_0}{2} (v - v_0) .$$

Hence

$$\frac{\partial H}{\partial p} (v,p;v_0,P_0) = \frac{\partial \hat{e}}{\partial p} + \frac{v - v_0}{2} .$$

But

$$\frac{\partial \hat{e}}{\partial p} = \frac{\partial \tilde{e}}{\partial \theta} \Big/ \frac{\partial p}{\partial \theta} = \frac{c_v(v - b)}{R}$$

and so

$$\frac{\partial H}{\partial p} = \frac{c_v(v - b)}{R} + \frac{v - v_0}{2} \to +\infty \quad \text{as} \quad \inf c_v(\theta) \to \infty .$$

Therefore we can solve $H(v,p;v_0,P_0) = 0$ for p as a function of v, say $p = h(v)$, if $\inf_{0<\theta<\infty} c_v(\theta)$ sufficiently large. Assuming that this is so, we have

$$\left(\frac{\partial \hat{e}}{\partial p} + \frac{v - v_0}{2}\right) \frac{dh}{dv} + \frac{d\hat{e}}{dv} + \frac{p + P_0}{2} = 0 .$$

We also have

$$\frac{\partial \hat{e}}{\partial v} = \frac{\partial \tilde{e}}{\partial v} + \frac{\partial \tilde{e}}{\partial \theta} \frac{\partial \theta}{\partial v}\Big|_p = \frac{a}{v^2} - \frac{c_v(v - b)}{R} \frac{\partial p}{\partial v}\Big|_\theta .$$

Thus

(3.6)
$$\frac{dh}{dv} = \left(\frac{c_v(v - b)}{R} \frac{\partial p}{\partial v}\Big|_\theta - \frac{a}{v^2} - \frac{p + P_0}{2}\right) \Big/ \left(\frac{v - v_0}{2} + \frac{c_v(v - b)}{R}\right) ,$$

and hence $\dfrac{dh}{dv} \to \dfrac{\partial p}{\partial v}\Big|_\theta$ uniformily on I as $\inf_{0<\theta<\infty} c_v(\theta) \to \infty$. Thus

$$h(v) - h(v_0) + \int_{v_0}^{v} p(z,\theta_0)dz = p(v,\theta_0) - P_0$$

uniformly on I as $\inf_{0<\theta<\infty} c_v(\theta) \to \infty$. Therefore $h(v) \to p(v,\theta_0)$ uniformly on I as $\inf_{0<\theta<\infty} c_v(\theta) \to \infty$, since $h(v_0) = p(v_0,\theta_0)$.

Let us define

(3.7)
$$\gamma = \frac{R + c_v}{c_v} .$$

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<u>Lemma 3.2.</u>

If c_v is constant, then $H(v,p;v_0,p_0) = 0$ can be solved algebraically for p as a function of v.

(3.8)
$$p\left\{\frac{(\gamma + 1)}{2} v - \frac{(\gamma - 1)}{2} v_0 - b\right\} + \frac{a}{v^2} \{(2 - \gamma)v - b\} =$$

$$p_0\left\{\frac{(\gamma + 1)}{2} v_0 - \frac{(\gamma - 1)}{2} v - b\right\} + \frac{a}{v_0^2} \{(2 - \gamma)v_0 - b\} .$$

Let $p = h(v)$ be the solution of $H(v,p;v_0,p_0) = 0$ then

(3.9) $\quad (v - v_s)\left\{h(v) + \frac{\gamma - 1}{\gamma + 1} p_0\right\} = \frac{2a}{\gamma + 1} \left\{\frac{2\gamma(v_0 - b)}{a(\gamma + 1)} p_0 + \frac{\gamma - 2}{v} + \frac{b}{v^2} - \frac{\gamma - 2}{v_0} - \frac{b}{v_0^2}\right\} ,$

where

(3.10)
$$v_s = \frac{(\gamma - 1)v_0 + 2b}{\gamma + 1} .$$

<u>Proof.</u>

We have $p = \frac{R\theta}{v - b} - \frac{a}{v^2}$, and hence

$$\theta = \frac{(v - b)}{R} \left(p + \frac{a}{v^2}\right)$$

$$\tilde{e} = c_v\theta - \frac{a}{v} = \frac{v - b}{\gamma - 1} \left(p + \frac{a}{v^2}\right) - \frac{a}{v} .$$

Now $H(v,p;v_0,p_0) = \tilde{e} - \tilde{e}_0 + \frac{p + p_0}{2} (v - v_0)$, and so the results follow.

It is clear from (3.9) that $h(v)$ has exactly one singularity at, $v = v_s$. As $c_v \uparrow \infty$, then $\gamma \downarrow 1$ and $v_s \downarrow b$; as $c_v \downarrow 0$, then $\gamma \uparrow \infty$ and $v_s \uparrow v_0$.

<u>Lemma 3.3.</u>

Suppose c_v is constant. Then

(3.11)
$$\lim_{v \to \infty} h(v) = -\frac{\gamma - 1}{\gamma + 1} p_0 .$$

<u>Proof.</u>

The result follows at once from formula (3.9).

In the next lemma we shall examine the behavior of $h(v)$ as v approaches v_s. To this end we define

$$(3.12) \qquad \sigma = \text{sgn}\left\{\frac{b^2}{a}P_0 + \frac{\gamma + 1}{\gamma((\gamma - 1)y + 2)^2}\left\{(\gamma - 1)(\gamma - 2) + \frac{4(\gamma - 1)}{y} + \frac{2}{y^2}\right\}\right\}$$

where $y = \dfrac{v_0}{b}$.

Lemma 3.4.

Suppose c_v is constant and $(v_0, p_0) \in \hat{\Omega}$. Then

$$(3.13) \qquad \lim_{v \uparrow v_s} h(v) = \begin{cases} -\infty, & \text{if } \sigma = 1 \\ +\infty, & \text{if } \sigma = -1 \end{cases},$$

$$(3.14) \qquad \lim_{v \downarrow v_s} h(v) = \begin{cases} +\infty, & \text{if } \sigma = 1 \\ -\infty, & \text{if } \sigma = -1 \end{cases},$$

$$(3.15) \qquad \lim_{v \to v_s} h(v) = \frac{-a}{\gamma + 1}\left\{\left(\frac{\gamma - 1}{a}\right)P_0 + \frac{(\gamma - 2)}{v_s^2} + \frac{2b}{v_s^3}\right\}, \quad \text{if } \sigma = 0 .$$

Lemma 3.5.

If c_v is constant and $\sigma = 1$ or 0 and $\gamma \geqslant 2$ or $v_0 < \dfrac{(2\gamma - 1)b}{(\gamma - 1)(2 - \gamma)}$, then

$$h(v) \leqslant -\frac{(\gamma - 1)}{(\gamma + 1)}P_0 \quad \text{for } b < v < v_s .$$

Proof.

Let

$$(3.16) \qquad f(v) = \frac{2\gamma(v_0 - b)}{a(\gamma + 1)}P_0 + \frac{\gamma - 2}{v} + \frac{b}{v^2} - \frac{\gamma - 2}{v_0} - \frac{b}{v_0^2} ,$$

so that the right hand side of equation (3.9) is given by $\dfrac{2a}{\gamma + 1}f(v)$. Lemma 3.4 now follows since $\sigma = \text{sign } f(v_s)$. To obtain Lemma 3.5, we have by hypothesis that $\sigma = 1$ or 0 and hence $f(v_s) \geqslant 0$. We may write (3.16) as,

$$(3.17) \qquad f(v) = f(v_s) + \frac{\gamma - 2}{v} + \frac{b}{v^2} - \frac{\gamma - 2}{v_s} - \frac{b}{v_s^2} .$$

If $\gamma > 2$, then $f(v) > 0$ for $b < v < v_s$. Now suppose that $1 < \gamma < 2$. The first and second derivatives of $f(v)$ are,

$$f'(v) = \frac{2 - \gamma}{v^2} - \frac{2b}{v^3} \quad \text{and} \quad f''(v) = -2\left(\frac{2 - \gamma}{v^3} - \frac{3b}{v^4}\right) .$$

Now $f'(v)$ has only one zero at $\bar{v} = \frac{2b}{2 - \gamma}$ and $f''(\bar{v}) = \frac{2b}{\bar{v}^{-4}} > 0$. So that $f(v)$ takes on its minimum value at $\bar{v}$. If $v_s < \bar{v}$ then $f(v) > f(v_s) > 0$ for $b > v < v_s$ since $\sigma = \text{sgn } f(v_s) = 1$ or 0 and $f(v)$ strictly decreasing on $b < v < \bar{v}$. But $v_s < \bar{v}$ if and only if $v_0 < \frac{(2\gamma - 1)b}{(\gamma - 1)(2 - \gamma)}$ by definition. Thus under the hypotheses of the lemma $f(v) > 0$ for $b < v < v_s$. Now (3.9) can be written as

$$(3.18) \qquad (v - v_s)\left\{h(v) + \frac{\gamma - 1}{\gamma + 1} p_0\right\} = \frac{2a}{\gamma + 1} f(v)$$

and thus

$$(3.19) \qquad h(v) < - \frac{\gamma - 1}{\gamma + 1} p_0 \quad \text{for} \quad b < v < v_s .$$

Lemma 3.6.

If the hypotheses of Lemma 3.5 are satisfied then the Rankine-Hugoniot jump conditions <u>cannot</u> be satisfied if $p_0 > 0$ and $v_1 < v_s$.

Proof.

In order for the Rankine-Hugoniot jump conditions to be satisfied, we must have

$$-m^2 = \frac{h(v_1) - p_0}{v_1 - v_0} ,$$

and so $h(v_1) - p_0 > 0$, since $v_1 < v_0$. But by Lemma 3.5 we have $h(v_1) - p_0 < \frac{-2\gamma p_0}{\gamma + 1} < 0$ and hence the conclusions follows.

SECTION 4.

<u>Theorem 4.1.</u>

Assume that $(v_1, p_1), (v_0, p_0) \in \hat{\Omega} \setminus \text{closure } (\hat{\Omega}_u)$ and that $v_1 < v_0$,

$H(v_1, p_1; v_0, p_0) = 0$, and

$$\frac{p_1 - p_0}{v_1 - v_0} = -m^2 < 0 .$$

Furthermore, suppose that the chord connecting (v_1, p_1) to (v_0, p_0) lies above the graph

of $H(v, p; v_0, p_0) = 0$ on the interval $v_1 < v < v_0$ and is not tangent to the graph at

either end point. Assume finally that the straight line extension of this chord does not

intersect the graph of $H(v, p; v_0, p_0) = 0$ when $v > v_0$ and $p > p(v, 0)$. Then there exists

a unique compression shock-layer connecting (v_0, p_0) to (v_1, p_1). Furthermore,

$\theta_1 > \theta_0$, $\tilde{n}_1 > \tilde{n}_0$, the flow is supersonic at the state (v_0, p_0), and subsonic at the

state (v_1, p_1).

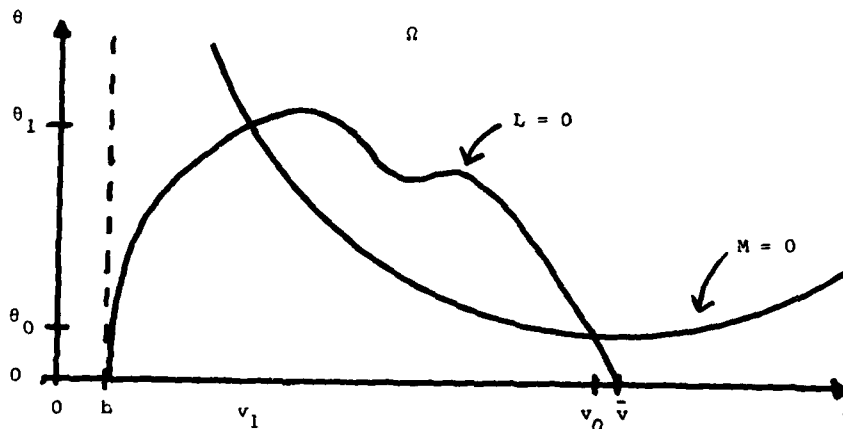The proof of Theorem 4.1 will be carried out with the aid of the following lemmas.

Figure 4

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<u>Lemma 4.2.</u>

Each of the equations $M(v,\theta) = 0$ and $L(v,\theta) = 0$ uniquely define θ as a function of v, say $\theta = \theta_M(v)$ and $\theta_L(v)$. Furthermore under the hypotheses of Theorem 4.1 the curve $L = 0$ intersects the v-axis in exactly two points $v = b$ and $v = \bar{v}$, where $\bar{v} > v_0$. Moreover the curve $L = 0$ lies above (below) the curve $M = 0$ in Ω when $v_1 < v < v_0$ ($v_0 < v < v_1$). That is

(4.1) $$\theta_L(v) > \theta_M(v) \quad \text{for} \quad v_1 < v < v_0$$

(4.2) $$\theta_L(v) < \theta_M(v) \quad \text{for} \quad v_0 < v < \bar{v}.$$

<u>Proof.</u>

We have

(4.3) $$\frac{\partial M}{\partial \theta} = c_v > 0$$

and

(4.4) $$\frac{\partial L}{\partial \theta} = \frac{\partial p}{\partial \theta} > 0$$

so that $M = 0$ and $L = 0$ can be solved for θ as a function of v. In particular

(4.5) $$\theta_L(v) = \frac{v - b}{R}\left(\frac{a}{v^2} + p_0 - m^2(v - v_c)\right).$$

It is clear from (4.5) that $\theta_L(b) = 0$. In the $v - p$ plane the line $L = 0$ intersects the lower boundary of the region $\hat{\Omega}$, given by $p = p(v,0)$, $b < v < \infty$, only once since $L = 0$ has negative slope and $p = p(v,0)$ has positive slope. Therefore in the $v - \theta$ plane the curve $L = 0$ intersects the v-axis (i.e. $\theta = 0$) only once in the interval (b,∞).

The curves $L = 0$ and $M = 0$ intersect in Ω if and only if the line $L = 0$ and the curve $H = 0$ intersect in $\hat{\Omega}$, as follows immediately from the relation

(4.6) $$H = M + \frac{1}{2}(v - v_0)L.$$

Thus it is sufficient to prove that the curve $L = 0$ lies above the curve $M = 0$ for at least one value of v in the interval (v_1, v_0). We have

(4.7) $$\frac{\partial H}{\partial p}\bigg|_v = \frac{c_v(v - b)}{R} + \frac{(v - v_0)}{2} > 0$$

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in some neighborhood of (v_0, p_0). Hence, by our assumption, in this neighborhood $H > 0$ on $L = 0$ if $v < v_0$. Thus by (4.6), in this neighborhood we also have $M > 0$ on $L = 0$ when $v < v_0$.

From Lemma 4.2 and equation (4.5) we see that the graph of $\theta_L(v)$ lies in Ω only when the domain of $\theta_L(v)$ is $(b, \bar{v})$. In the next lemma we shall show that the graph of $\theta_M(v)$ lies in Ω when $b < v < \infty$.

Lemma 4.3.

Under the hypotheses of Theorem 4.1 there is a constant $\bar{\theta} > 0$ such that $\theta_M(v) > \bar{\theta}$ for all $v > b$, with equality holding only at $v = \bar{v}$.

Proof.

The curve $M = 0$ lies above the curve $L = 0$ for $v_0 < v < \bar{v}$ by Lemma 4.2, and the curve $L = 0$ lies above the v-axis for $v_0 < v < \bar{v}$. It is sufficient to show that the curve $\theta = \theta_M(v)$ takes on its minimum value in the interval (b, ∞) at $v = \bar{v}$. Now

$$(4.8) \qquad \frac{\partial M}{\partial v} = \theta \frac{\partial p}{\partial \theta} - L$$

and by (4.3), since $c_v = c_v(\theta)$,

$$(4.9) \qquad \frac{\partial^2 M}{\partial \theta \partial v} = 0$$

so that

$$(4.10) \qquad \frac{\partial M}{\partial v}(v, \theta) = \frac{\partial M}{\partial v}(v, 0) = -L(v, 0) \ .$$

From (4.5) and Lemma 4.2 we see that $L(v, 0) < 0$ when $b < v < \bar{v}$ and $L(v, 0) > 0$ when $\bar{v} < v < \infty$. But

$$(4.11) \qquad \frac{d\theta_M(v)}{dv} = -\frac{\partial M}{\partial v}(v, 0) \Big/ \frac{\partial M}{\partial \theta}(v, \theta) = \frac{L(v, 0)}{c_v(\theta)}$$

and so $\theta_M(v)$ takes on it minimum at $v = \bar{v}$ and $\theta_M(v) > \theta_M(\bar{v})$ for all $v > b$, with equality holding only at $v = \bar{v}$.

Lemma 4.4.

Under the hypotheses of Theorem 4.1 the critical point $(v_1, 0, \theta_1)$ is a saddle point with a one-dimensional stable manifold and a two dimensional unstable manifold.

Furthermore the fluid velocity is subsonic at the back state (v_1, θ_1) and is supersonic at the front state (v_0, θ_0).

Proof.

The acoustic speed c is given by

$$(4.12) \qquad c^2 = \frac{\partial p}{\partial \rho}\Big|_{\tilde{\eta}} = -v^2 \frac{\partial p}{\partial v}\Big|_{\tilde{\eta}} .$$

Thus

$$(4.13) \qquad u^2 - c^2 = v^2 \left(\frac{\partial p}{\partial v}\Big|_{\tilde{\eta}} + m^2\right)$$

since $u = mv$. We need to show that

$$(4.14) \qquad \frac{\partial p}{\partial v}\Big|_{\tilde{\eta}} + m^2 > 0 \quad \text{at} \quad (v_0, \theta_0)$$

and

$$(4.15) \qquad \frac{\partial p}{\partial v}\Big|_{\tilde{\eta}} + m^2 < 0 \quad \text{at} \quad (v_1, \theta_1) .$$

Now by hypotheses the chord connecting (v_1, θ_1) to (v_0, θ_0) has slope $-m^2$ and lies above the Hugoniot curve ($H = 0$ curve) in $\hat{\Omega}$. Thus the slope of the curve $H = 0$ is greater than the slope of the chord at (v_0, θ_0).

From (2.28) and the Gibbs relation we have

$$(4.16) \qquad dH = \left(\theta \frac{\partial \tilde{\eta}}{\partial p}\Big|_v + \frac{v - v_0}{2}\right) dp + \left(\theta \frac{\partial \tilde{\eta}}{\partial v}\Big|_p - \frac{p - p_0}{2}\right) dv .$$

Thus

$$(4.17) \qquad \frac{dp}{dv}\Big|_H = -\frac{\partial \tilde{\eta}}{\partial v}\Big|_p \Big/ \frac{\partial \tilde{\eta}}{\partial p}\Big|_v = \frac{\partial p}{\partial v}\Big|_{\tilde{\eta}} \quad \text{at} \quad (v_0, p_0)$$

and so

$$\frac{\partial p}{\partial v}\Big|_{\tilde{\eta}} + m^2 > 0 \quad \text{at} \quad (v_0, p_0) .$$

This proves (4.14). We next show (4.15).

By Lemma 4.2 the curve $L = 0$ lies below the curve $M = 0$ for $v_0 < v < \bar{v}$ and so

$$(4.18) \qquad \frac{d\theta_L}{dv}(v_1) > \frac{d\theta_M}{dv}(v_1) .$$

The strictness of the inequality can be shown to follow from the non-tangency hypothesis and equation (4.4). Now

$$(4.19) \qquad \frac{d\theta_L}{dv} = -\frac{\partial L}{\partial v} \Big/ \frac{\partial L}{\partial \theta}$$

and

(4.20)
$$\frac{d\theta_M}{dv} = -\frac{\partial M}{\partial v} \Big/ \frac{\partial M}{\partial \theta}$$

thus by (4.18)

(4.21)
$$D \equiv \frac{\partial L}{\partial v} \frac{\partial M}{\partial \theta} - \frac{\partial M}{\partial v} \frac{\partial L}{\partial \theta} < 0 \quad \text{at} \quad (v_1, \theta_1)$$

since $\frac{\partial L}{\partial \theta}$ and $\frac{\partial M}{\partial \theta}$ are positive. Substituting (4.3), (4.4), (4.8), and

(4.22)
$$\frac{\partial L}{\partial v} = \frac{\partial p}{\partial v} + m^2$$

into (4.21) gives

(4.23)
$$D = c_v \left(\frac{\partial p}{\partial v} + m^2 \right) - \left(\theta \frac{\partial p}{\partial \theta} - L \right) \frac{\partial p}{\partial \theta} \quad \text{at} \quad (v_1, \theta_1) \ .$$

But $L(v_1, \theta_1) = 0$ and (see Section 2)

(4.24)
$$\frac{\partial p}{\partial v} - \frac{\theta}{c_v} \left(\frac{\partial p}{\partial \theta} \right)^2 = \frac{\partial p}{\partial v} \Big|_{\tilde{\eta}}$$

so that

(4.25)
$$D = c_v \left(\frac{\partial p}{\partial v} \Big|_{\tilde{\eta}} + m^2 \right) \ .$$

Since $D < 0$, condition (4.15) follows at once.

To show that $(v_1, 0, \theta_1)$ is a saddle point we linearize (2.24) about $(v_1, 0, \theta_1)$ to obtain

(4.26)
$$\begin{pmatrix} v' \\ w' \\ \theta' \end{pmatrix} = A \begin{pmatrix} v - v_1 \\ w \\ \theta - \theta_1 \end{pmatrix}$$

where

(4.27)
$$A = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{\sigma} \frac{\partial L}{\partial v} & m \frac{\lambda + 2\mu}{\sigma} & -\frac{1}{\sigma} \frac{\partial L}{\partial \theta} \\ \frac{m}{\kappa} \frac{\partial M}{\partial v} & 0 & \frac{m}{\kappa} \frac{\partial M}{\partial \theta} \end{pmatrix} \ ,$$

the entries being evaluated at (v_1, θ_1).

Now at (v_1, θ_1)

(4.28)
$$\det A = \frac{m}{\kappa \sigma} \left(\frac{\partial L}{\partial v} \frac{\partial M}{\partial \theta} - \frac{\partial L}{\partial \theta} \frac{\partial M}{\partial v} \right) = \frac{m}{\kappa \sigma} D < 0$$

by (4.21). Also

(4.29)
$$\text{trace } A = m\left(\frac{\lambda + 2\mu}{\sigma} + \frac{c_v}{\kappa}\right) > 0 .$$

Let ρ_1, ρ_2 and ρ_3 be the eigenvalues of A. Suppose first that these are all real and that $\rho_1 < \rho_2 < \rho_3$. Then $\rho_1\rho_2\rho_3 = \det A < 0$ by (4.28) and so either $\rho_1 < 0$, $\rho_2 < 0$, and $\rho_3 < 0$ or $\rho_1 < 0$, $\rho_2 > 0$, and $\rho_3 > 0$. But $\rho_1 + \rho_2 + \rho_3 = \text{trace } A > 0$ by (4.29), and hence $\rho_1 < 0$, $\rho_2 > 0$ and $\rho_3 > 0$. Next suppose that ρ_1, ρ_2 and ρ_3 are not all real. Let $\rho_1 = \alpha - i\beta$ and $\rho_2 = \alpha + i\beta$, where α, β and ρ_3 are all real. Then $(\alpha^2 + \beta^2)\rho_3 = \det A < 0$; hence $\rho_3 < 0$ and $2\alpha + \rho_3 = \text{trace } A > 0$, so $\alpha > 0$. Therefore the critical point $(v_1, 0, \theta_1)$ is a saddle point with a one-dimensional stable manifold and a two-dimensional unstable manifold.

Define
(4.30)
$$\Phi(v, w, \theta) = m\left(\tilde{\eta} - \tilde{\eta}_1 + \frac{\sigma w^2}{2\theta} - \frac{M}{\theta}\right) .$$

Lemma 4.5.

Let $(v(x), w(x), \theta(x))$ be any solution of (2.24). Then
(4.31)
$$\frac{d\Phi}{dx}(v(x), w(x), \theta(x)) > 0 .$$

That is Φ is nondecreasing along the trajectories of (2.24).

Proof.

We can use (2.13), (2.23), and (2.24c) to write (4.30) as
(4.32)
$$\Phi = m(\eta - \tilde{\eta}_1) - \frac{\kappa\theta'}{\theta} .$$

But
(4.33)
$$\Phi' = m\eta' - \left(\frac{\kappa\theta'}{\theta}\right)' > 0$$

by the Clausius-Duhem inequality, completing the proof.

Now $\Phi(v_1, 0, \theta_1) = 0$ and
(4.33)
$$\frac{\partial\Phi}{\partial\theta}(v, 0, \theta) = \frac{m}{\theta^2} M(v, \theta)$$

so that $\frac{\partial\Phi}{\partial\theta}(v_1, 0, \theta) > 0$ (< 0) for $\theta > \theta_1$ $(\theta < \theta_1)$. Hence
(4.34)
$$\Phi(v_1, 0, \theta) > 0 \quad \text{for} \quad \theta \neq \theta_1 .$$

The function $\theta_L(v)$ has a maximum on the interval $(b, \bar{v})$, say $\bar{\theta}$, since by (4.5) we see that $\theta_L(b) = \theta_L(\bar{v}) = 0$ and $\theta_L(v) > 0$ for $b < v < \bar{v}$. Now

(4.35)
$$\frac{\partial \phi}{\partial v}(v,0,\theta) = \frac{m}{\theta}L(v,\theta)$$

and so

(4.36)
$$\frac{\partial \phi}{\partial v}(v,0,\bar{\bar{\theta}}) > 0$$

since $L > 0$ to the right of the line $v = \bar{v}$. Thus

(4.37)
$$\phi(v,0,\bar{\bar{\theta}}) > 0 \quad \text{for} \quad v > v_0$$

since $\phi(v_1,0,\bar{\bar{\theta}}) > 0$. Equality occurs if $\theta_1 = \bar{\bar{\theta}}$).

We next show that the curve $M = 0$ intersects the line $\theta = \bar{\bar{\theta}}$ for some $\bar{v} > \bar{v}$. Now from (4.11) we have

(4.38)
$$\frac{d\theta_M}{dv}(v) = \frac{L(v,0)}{c_v} = \frac{1}{c_v}\left(m^2(v - v_0) - P_0 - \frac{a}{v^2}\right)$$

and so by the definition of $\bar{v}$,

(4.39)
$$\frac{d\theta_M}{dv}(\bar{v}) = 0 .$$

Therefore, $\theta_M(v) \uparrow \infty$ as $v \uparrow \infty$ since by (4.38) $\frac{d\theta_M}{dv}(v) \uparrow \infty$ as $v \uparrow \infty$. Thus the curve $M = 0$ intersects the line $\theta = \bar{\bar{\theta}}$ for some $\bar{\bar{v}} > \bar{v}$. From (4.38) and (4.33) it is clear that $\phi(\bar{v},0,\theta) > 0$ for $0 < \theta < \bar{\bar{\theta}}$.

Put

(4.40)
$$\bar{w} = \sqrt{\max_{\substack{v_1 \leq v \leq \bar{\bar{v}} \\ \bar{\theta} \leq \theta \leq \bar{\bar{\theta}}}} -2\theta\phi(v,0,\theta)}$$

We define a box B in phase space by

(4.41)
$$B = \{(v,w,\theta); v_1 < v < \bar{\bar{v}}, -\bar{w} < w < \bar{w}, \bar{\theta} < \theta < \bar{\bar{\theta}}\} .$$

Note that with the exception of the bottom of the box and the point $(v_1,0,\theta_1)$ we have $\phi > 0$ on ∂B.

Lemma 4.6.

Under the hypotheses of Theorem 4.1 every trajectory which intersects the bottom of the box B leaves the box.

Proof.

We have from (2.24c)

$$\kappa\theta' = m\left(-\frac{1}{2}\frac{\partial\theta\sigma}{\partial\theta}w^2 + M(v,\theta)\right) .$$

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From Lemma 4.3 the curve $M = 0$ *does not drop below the line* $\theta = \bar{\theta}$ *in* Ω. But $\frac{\partial M}{\partial \theta} = c_v > 0$ and so

(4.42) $$M(v,\theta) < 0 \quad \text{for} \quad b < v < \infty \quad \text{and} \quad 0 < \theta < \bar{\theta}$$

with equality holding only if $(v,\theta) = (\bar{v},\tilde{\theta})$. Now

(4.43) $$\frac{\partial \theta \sigma}{\partial \theta} = \sigma + \frac{\theta \partial \sigma}{\partial \theta}$$

and by hypothesis $\sigma > 0$ and

(4.44) $$\lim_{\theta \downarrow 0} \inf \frac{\theta \partial \sigma}{\partial \theta} (v,\theta) > 0 \quad \text{for} \quad b < v < \infty$$

so that

(4.44) $$\frac{\partial \theta \sigma}{\partial \theta} > 0 \quad \text{for} \quad b < v < \infty$$

and θ sufficiently small and positive. We can choose $\bar{\theta}$ smaller if necessary so that (4.44) holds when $0 < \theta < \bar{\theta}$. Thus $\theta' < 0$ on the bottom of the box B with the possible exception of $(\bar{v},0,\tilde{\theta})$. Therefore every trajectory which intersects the bottom of the box at some point leaves the box at that point.

Lemma 4.7.

Under the hypotheses of Theorem 4.1 one of the trajectories of the stable manifold of $(v_1,0,\theta_1)$ enters the box B while the other never does. Thus there can be at most one trajectory of (2.24) connecting $(v_0,0,\theta_0)$ to $(v_1,0,\theta_1)$.

Proof.

We need to show that the line tangent to the stable manifold is transverse to the plane $v = v_1$ in phase space. This line is parallel to the *eigenvector* associated with the negative eigenvalue ρ of (4.27). Let $(\xi_1,\xi_2,\xi_3)^T$ be this eigenvector. Then we have

(4.44) $$A \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \rho \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

We assert that $\xi_1 \neq 0$, which is the required transversality condition. Suppose for contradiction that $\xi_1 = 0$. Then from (4.44) and (4.27) we have

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$$\xi_2 = \rho\xi_1 = 0$$

(4.45)

$$(\kappa\rho - mc_v)\xi_3 = m\frac{\partial M}{\partial v}\xi_1 = 0 .$$

But $\kappa\rho - mc_v < 0.$ Hence $\xi_1 = \xi_2 = \xi_3 = 0,$ which is impossible.

Now suppose one of the trajectories forming the stable manifold of $(v_1,0,\theta_1)$ crosses the plane $v = v_1$ at some point. Then $\Phi > 0$ at this point since

(4.46)
$$\Phi(v,w,\theta) = \Phi(v,0,\theta) + \frac{m\sigma w^2}{2\theta} .$$

and $\Phi(v_1,0,\theta) > 0$ for $\theta \neq \theta_1$ by (4.34). By Lemma 4.5, Φ is nondecreasing along trajectories of (2.24), and so $\Phi(v_1,0,\theta_1) > 0.$ But $\Phi(v_1,0,\theta_1) = 0$ and hence any trajectory forming the stable manifold cannot cross the plane $v = v_1.$

If we replace x by $-x$ in the system of ordinary differential equaiton (2.24) then the direction of each trajectory in (2.24) wil be reversed. We shall refer to (2.24) with x replaced by $-x$ as the reversed system. Note that Φ decreases along any trajectory of the reversed system and hence Φ is a Liapunov function in the sense of LaSalle for the reversed system.

Let us denote by $\mathcal{O}$ the trajectory of the reversed system that leaves $(v_1,0,\theta_1)$ and enters B.

Lemma 4.8.

Under the hypotheses of Theorem 4.1 the trajectory $\mathcal{O}$ is bounded.

Proof.

We shall show that $\mathcal{O}$ is contained in B. For the reversed system, Φ is nonincreasing along $\mathcal{O}$ and hence $\Phi \leq 0$ on $\mathcal{O}$ since $\Phi(v_1,0,\theta_1) = 0.$ But $\Phi > 0$ on ∂B with the exception of the point $(v_1,0,\theta_1)$ and the bottom face of B. Thus $\mathcal{O}$ cannot cross ∂B except possibly at the bottom face. But by Lemma 4.6 and the fact that the system is reversed, every trajectory which intersects the bottom face enters B. Thus $\mathcal{O}$ cannot cross ∂B and so it is bounded.

To complete the proof of Theorem 4.1 we need to show that $\mathcal{O}$ enters $(v_0,0,\theta_0)$, that is $\omega(\mathcal{O}) = \{(v_0,0,\theta_0)\}$ for the reversed system. Since Φ is a Liapunov function in the sense of LaSalle for the reversed system and $\mathcal{O}$ is bounded, $\omega(\mathcal{O})$ is contained in the largest invariant subset of $S \equiv \{(v,w,\theta) \in \text{closure of } B : \Phi' = 0\}$, by LaSalle's invariance principle [12].

From (4.33) we have

(4.47)
$$\Phi' = m\eta' - \frac{(\kappa\theta')'}{\theta} + \frac{\kappa\theta'^2}{\theta^2}$$

Now upon substituting (2.24c) and (2.13) into (4.47) we have

(4.48)
$$\Phi' = \frac{m^2}{\theta}(\lambda + 2\mu)w^2 + \frac{\kappa\theta'^2}{\theta^2} .$$

Therefore $\Phi' = 0$ implies $w = \theta' = 0$ and by (2.24) also $\sigma w' + L = 0$ and $M = 0$ when $\Phi' = 0$. Thus the largest invariant subset of S is $\{(v_1,0,\theta_1),(v_0,0,\theta_0)\}$. But $\theta' \neq 0$ on $\mathcal{O}$ so that $\Phi' < 0$ somewhere on $\mathcal{O}$ and hence $\Phi < 0$ on $\mathcal{O}$. Therefore $\omega(\mathcal{O}) = \{(v_0,0,\theta_0)\}$ and so the trajectory $\mathcal{O}$ connects $(v_1,0,\theta_1)$ to $(v_0,0,\theta_0)$.

It remains to show that $\tilde{\eta}_1 > \tilde{\eta}_0$ and $\theta_1 > \theta_0$. The first follows from (2.8), since (4.30) and (4.48) show that the equality cannot hold. The second follows from (4.11) since $L(v,0) < 0$ for $v_1 < v < v_0$.

Example 1.

Suppose that c_v is a large constant, so that by Lemma 3.1 the Hugoniot curve generated by (v_0,p_0) is near the isotherm $p(v,\theta_0)$. Let us suppose that (v_0,p_0) is in the vapor region and (v_1,p_1) is in the liquid region. Furthermore, suppose that the straight line through (v_0,p_0) and (v_1,p_1) intersects the Hugoniot curve at only these two points (see Figure 5).

In this example the hypotheses of Theorem 4.1 are satisfied and hence there is a shock connecting (v_0,p_0) to (v_1,p_1). The shock layers converts vapor in the equilibrium state (v_0,p_0) into liquid in the equilibirum state (v_1,p_1). We shall call such a layer a liquefaction layer [13].
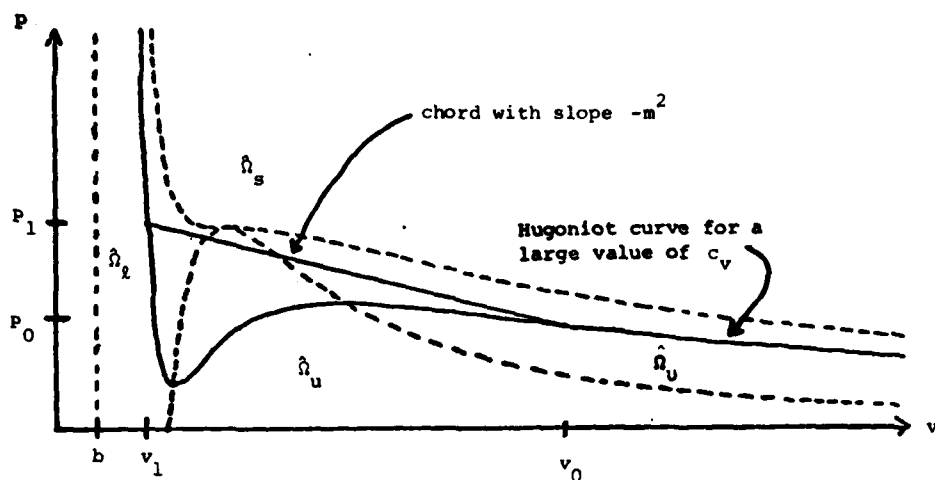
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Figure 5

Example 2.

We keep the same set up as in Example 1, but with (v_1, p_1) also in the vapor region and $v_1 < v_0$ (see Figure 6). Then there is a gas-gas compressive shock layer converting gas in the more rarefied equilibrium state (v_0, p_0) to gas in the hotter denser equilibrium state (v_1, p_1).
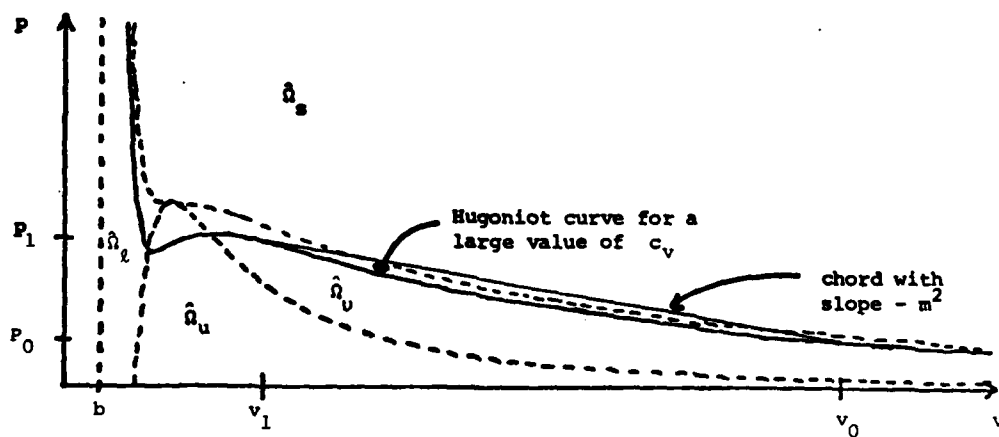

Figure 6

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Example 3.

Let us suppose that c_v is constant and $0 < c_v < R$, so that $\gamma > 2$. Let us choose (v_0, p_0) in either the vapor or super-heated vapor region, with $v_0 > 7b$ and $p_0 > 0$. Then $\sigma = 1$ by (3.12), and thus the only reachable states on the Hugoniot curve are to the right of

$$v_s = \frac{(\gamma - 1)v_0 + 2b}{\gamma + 1}$$

by Lemma 3.6. Since $v_0 > 7b$ and $\gamma > 2$ we have

$$v_s > \frac{(\gamma - 1)7b + 2b}{\gamma + 1} = 7b - \frac{12b}{\gamma + 1} > 3b .$$

But the liquid region lies to the left of $v = 3b$, and hence no liquification shock is possible (see Figure 7).
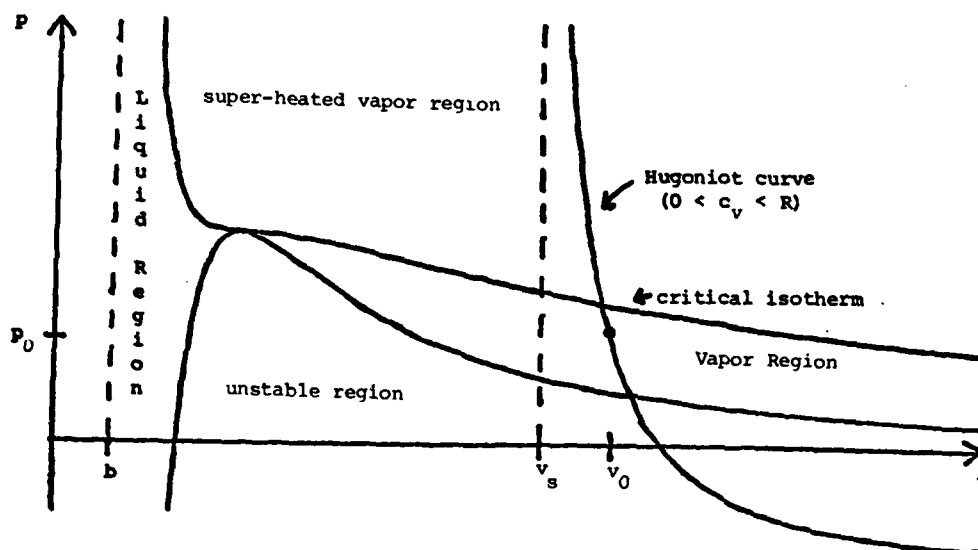

Figure 7

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    This paper gives sufficient conditions to guarantee the existence of a shock layer solution connecting two different equilibrium states in a van der Waals fluid. In particular, the equilibrium states can belong to two different phases of the fluid. The constitutive laws come from a modified Korteweg theory which is compatible with the Clausius Duhem inequality. The Clausius Duhem inequality in turn gives rise to a Liapunov function. The main mathematical tool is the LaSalle invariance principle.